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Algebraic Bethe ansatz with boundary condition for $SU_{p,q}(2)$ invariant spin chain

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Abstract. We have analysed a generalized Heisenberg spin chain invariant under the quantum group $SU_{p,q}(2)$. Instead of the usual periodic boundary condition, non-trivial boundary conditions are imposed *a la* Sklyanin. *R*-matrix and Bethe eigenstates are constructed explicitly. The *R*-matrix describes a vertex type model which can be shown to be connected to the *xxz* type six vertex model under the Akutsu–Deguchi–Wadati-type symmetry-breaking transformation.

1. Introduction

Quantization of nonlinear integrable system is an important problem, and has been studied exhaustively over the last two decades [1]. Heisenberg spin chain occupies a central position in such studies [2]. Importance is enhanced further due to the close analogy of the original treatment of Baxter [3] and the quantum inverse formulation due to Faddeev *et al* [4]. Lately, several new interesting features have come to light due to the use of the elegant formalism of the quantum group [5]. In recent communications the use of quantum groups in relation to the Heisenberg spin chain have been elegantly discussed by Karowski [6], Mezinescu *et al* [7] and Pasquier *et al* [8]. Here in this communication we consider a two-parameter quantum group $S_{p,q}U(2)$ and discuss the corresponding generalized spin chain by obtaining the R-matrix and the Bethe eigenstates. Our treatment emphasizes the role of non-periodic boundary conditions at the ends, following the earlier treatment of Sklyanin [9]. It is interesting to note that the Hamiltonian for the system under consideration is a simple generalization of the $SU_q(2)$-spin chain discussed earlier by Sklyanin. It is also observed that the present model can be connected to the usual *xxz* type model by a symmetry-breaking transformation originally proposed by Akutsu *et al* [10]. As a consequence it is seen to follow that the Hamiltonian describing the present model can also be transformed to that of the gauge transformed *xxz* Hamiltonian in the sense of de Vega-Lopes [11].

2. Formulation

The $SU_{p,q}(2)$ is a quantum group obtained by the two-parameter deformation of the basic Lie algebra SU(2) and is governed by the following relations between the generators

X_+, X_-, and H

$$[H, X_\pm] = \pm X_\pm$$

$$[X_+, X_-]_{pq} = X_+X_- - pq^{-1}X_-X_+ = [2H]_{pq} \tag{1}$$

where

$$[X]_{pq} = \frac{q^x - p^{-x}}{q - p^{-1}}.$$

We consider the L-operator

$$L_n(\lambda) = \begin{bmatrix} \lambda q^{H_n} - \lambda^{-1} q^{H_n} & (q-p^{-1})X_n^- \\ (q-p^{-1})X_{n^+} & \lambda q^{-H_n} - {}^{-1}p^{H_n} \end{bmatrix} \tag{2}$$

with the usual notation, $L_n^1 = L_n(\lambda) \otimes 1$ and $L_n^2(\mu) = 1 \otimes L_n(\mu)$ and the R-matrix defined via the condition

$$R(\lambda, \mu) L_n^1(\lambda) L_n^2(\mu) = L_n^2(\mu) L_n^1(\lambda) R(\lambda, \mu). \tag{3}$$

One immediately observes that the solution to the R-matrix can be written as

$$R = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b_1 & c & 0 \\ 0 & c & b_2 & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \tag{3a}$$

where

$$a = \frac{\lambda}{\mu} q - \frac{\mu}{\lambda} p^{-1} \qquad c = q - p^{-1} \qquad b_1 = \frac{\lambda}{\mu} - \frac{\mu}{\lambda} \qquad b_2 = p^{-1} q \left(\frac{\lambda}{\mu} - \frac{\mu}{\lambda} \right)$$

It is important to point out that the R-matrix is less symmetric than the $SU_q(2)$ case.

In the following it will be easier to use; $\lambda = e^{u_1}$, $\mu = e^{u_2}$ and $u_1 - u_2 = u$. To proceed further, let us note the following properties of the R-matrix,

(a) $$p_{12} R_{12}(u) p_{12} = R_{21}(u) \tag{4}$$

where p_{12} is the permutation operator

(b) $$R_{12}(u) R_{21}(-u) = -4(p^{-1}q) \sinh(u+r) \sinh(u-r) I \tag{5}$$

where I is the unit matrix and $e^r = (pq)^{1/2}$, this being the unitarity.

(c) Crossing unitarity.

Furthermore, the R-matrix satisfies

$$R_{12}^{t_2}(u) R_{21}^{t_1}(-u-2r) = -4(p^{-1}q) \sinh u \sinh(u+2r) I. \tag{6}$$

Let us now consider two diagonal matrices $K_+(u)$ and $K_-(u)$ specifying the boundary conditions at the two ends of the spin chain, following the two conditions

$$R(u_1-u_2) K_-(u) R(u_1+u_2-r) K_-^2(u) = K_-^2(u_2) R_-(-u_1-u_2-r) K_-(u_1) R(u_1-u_2) \tag{7}$$

and

$$R(-u_1+u_2) K_-^{1t_1}(u_1) R(-u_1-u_2-r) K_+^{2t_2}(u_2)$$
$$= K_+^{2t_2}(u_2) R(-u_1-u_2-r) K_+^{1t_2}(u_1) R(-u_1+u_2). \tag{8}$$

It is not very difficult to observe that the solitons of (7) and (8) for K_+ and K_- can be written as

$$K_+=\begin{pmatrix}\sinh(u-r/2+\xi_+) & 0\\ 0 & \sinh(-u-r/2+\xi_+)\end{pmatrix}$$
$$K_-=\begin{pmatrix}\sinh(u-r/2+\xi_-) & 0\\ 0 & \sinh(-u-r/2+\xi_-)\end{pmatrix}. \tag{9}$$

Following Sklyanin we note that the Hamiltonian is generated by the commuting family

$$t(u)=\operatorname{Tr} K_+(u)\tau(u)$$

with

$$\tau(u)=T(u)K_-(u)\sigma_2T^{-1}(-u)\sigma_2$$

Let

$$\tau(u)=\begin{pmatrix}A(u) & B(u)\\ C(u) & D(u)\end{pmatrix} \tag{10}$$

which obeys

$$R_{12}(u_1-u_2)T^1(u)R_{21}(u_1+u_2-r)T^2(u)=T^2(u)R_{12}(u_1+u_2-r)T^1(u)R_{21}(u_1-u_2) \tag{11}$$

from which we obtain

$$\begin{aligned}A(u_1)B(u_2)={}&\frac{a(u_-)b_1(u_+-r)}{b_1(u_-)a(u_+-r)}B(u_2)A(u_1)-\frac{c(u_+-r)}{a(u_+-r)}B(u_1)D(u_2)\\&-\frac{c(u_-)b_1(u_+-r)}{b_1(u_-)a(u_+-r)}B(u_1)A(u_2)\end{aligned} \tag{12}$$

$$\begin{aligned}D(u_1)B(u_2)={}&\frac{a(u_-)}{b_1(u_+-r)b_2(u_-)}\left[a(u_+-r)-\frac{c^2(u_+-r)}{a(u_+-r)}\right]\\&\times B(u_2)D(u_1)-\left[\frac{c(u_-)a(u_+-r)}{b_1(u_+-r)b_2(u_-)}+\frac{a(u_-)c^2(u_+-r)}{a(u_+-r)b_1(u_+-r)}\times\frac{1}{b_2(u_-)}\right]\\&\times B(u_1)D(u_2)+\frac{c(u_+)}{b_1(u_-)b_2(u_-)}\times\left[\frac{a^2(u_-)}{a(u_+)}+c(u_-)c(-u_-)\right]B(u_1)A(u_2)\\&+B(u_2)A(u_1)\times\left[\frac{a(u_-)c(u_-)c(u_+-r)}{b_1(u_-)b_2(u_-)a(u_+-r)}+\frac{c(u_+-r)c(u_-)c(-u_-)}{b_1(u_-)b_2(-u_-)a(u_+-r)}\right]\end{aligned}$$

where

$$u_+=u_1+u_2;\ u_-=u_1-u_2. \tag{13}$$

It is now important to note that the R-matrix at $\lambda/\mu=-r$ is a projection operator, that is

$$R(-r)=\{(pq)^{1/2}-(pq)^{-1/2}\}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & -1 & (p^{-1}q)^{1/2} & 0\\ 0 & (p^{-1}q)^{1/2} & -(p^{-1}q) & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

and the matrix

$$p_{12}=-\frac{1}{(1+p^{-1}q)}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & -1 & (p^{-1}q)^{1/2} & 0\\ 0 & (p^{-1}q)^{1/2} & -(p^{-1}q) & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{14}$$

has the property that $p_{12}^2=p_{12}$. Then let $\tilde{\tau}(\mu)$ be the algebraic adjoint of $\tau(v)$ defined by;

$$\begin{aligned}\tilde{\tau}(u)&=(p+p^{-1}q)\,\mathrm{tr}_2\,p_{12}^{-}\tau^2(u)R_{21}(2u)\\ &=-\begin{pmatrix}b_1(2u)D+(p^{-1}q)^{1/2}c(2u)A & (p^{-1}q)^{1/2}a(2u)B\\ a(u)C(u)(p^{-1}q)^{1/2} & (p^{-1}q)^{1/2}c(2u)D(u)-p^{-1}qb_2(2u)A(u)\end{pmatrix}\\ &=\begin{pmatrix}\bar{D}(u) & -\bar{B}(u)\\ -\bar{c}(u) & \bar{A}(u)\end{pmatrix}\quad\text{(say).}\end{aligned} \tag{15}$$

so the new sets of commutation rules which are convenient for diagonalization are

$$\begin{aligned}A(u_1)B(u_2)&=\frac{(g-p^{-1})(q^{-1}\,\mathrm{e}^{2u_2}-p\,\mathrm{e}^{-2u_2})}{4\sinh(u_1-u_2)\sinh(2u_2)}B(u_1)A(u_2)\\ &\quad-\frac{(g-p^{-1})}{4(p^{-1}q)\sinh(u_1+u_2)\sinh(2u_2)}B(u_1)\bar{D}(u_2)\\ &\quad+\frac{\sinh(u_1-u_2-r)\sinh(u_1+u_2-r)}{\sinh(u_1+u_2)\sinh(u_1-u_2)}B(u_2)A(u_1)\end{aligned} \tag{16}$$

and

$$\begin{aligned}D(u_1)B(u_2)&=\frac{\{q\mathrm{e}^{u_1-u_2}-p^{-1}\,\mathrm{e}^{u_2-u_1}\}\{p\,\mathrm{e}^{u_1+u_2}-q^{-1}\,\mathrm{e}^{-u_1-u_2}\}}{4\sinh(u_1+u_2)\sinh(u_1-u_2)}\\ &\quad\times B(u_2)\bar{D}(u_1)-\frac{(q-p^{-1})p\,\mathrm{e}^{2u_1}-q^{-1}\,\mathrm{e}^{-2u_1}}{4\sinh(2u_2)\sinh(u_1-u_2)}B(u_1)\bar{D}(u_2)\\ &\quad+\frac{(p^{-1}q)^{1/2}(q-p^{-1})(p\,\mathrm{e}^{2u}-g^{-1}\,\mathrm{e}^{-2u})}{4\sinh(u_1+u_2)\sinh(2u_2)}(p^{-1}\,\mathrm{e}^{2u_2}-q\,\mathrm{e}^{-2u_2})\\ &\quad\times B(u_2)A(u_1).\end{aligned} \tag{17}$$

Finally

$$\begin{aligned}t(u)&=\mathrm{Tr}[K_+(u)\tau(u)]\\ &=\sinh(u+r/2+\xi_+)A(u)+\sinh(\xi_+-u-r/2)D(u)\\ &=\frac{\sinh(2u+r)\sinh(\alpha+u)}{\sinh(2u)}A(u)+\frac{\sinh(\alpha-u)}{2p^{-1}q\sinh(2u)}\bar{D}(u)\end{aligned} \tag{18}$$

where $\alpha=-r/2$.

3. The Bethe eigenstates

The Bethe eigenstates are now constructed by starting from the vacuum state defined by

$$\begin{aligned} A(u)|0\rangle &= \alpha(u)|0\rangle \\ D(u)|0\rangle &= d(u)|0\rangle \\ C(u)|0\rangle &= 0 \end{aligned} \tag{19}$$

the N excited state is

$$|N\rangle = B(\lambda_1)B(\lambda_2)\dots B(\lambda_N)|0\rangle$$

leading to the condition

$$\frac{\alpha(\lambda_n)}{d(\lambda_b)} = \frac{2p^{-1}q\sinh(2\lambda_n - r)\sinh(\alpha+\lambda_n)}{\sinh(\alpha-\lambda_n)} \times \prod_{j=1}^{n} \frac{\sinh(\lambda_n+\lambda_j-r)\sinh(\lambda_n-\lambda_j-r)}{\sinh(\lambda_n+\lambda_j+r)\sinh(\lambda_n+\lambda_j+r)} \tag{20}$$

by making the unwanted terms vanish. The energy eigenvalue for the corresponding state is obtained as

$$E_n = \frac{1}{2p^{-1}q}\frac{\sinh(\alpha-u)}{\sinh 2u}\prod_{n=1}^{N}\frac{\sinh(u-\lambda_n+r)\sinh(u+\lambda_n+r)}{\sinh(u+\lambda_n)\sinh(u-\lambda_n)} + \frac{\sinh(2u+r)\sinh(\alpha+u)}{\sinh(2u)} \times \prod_{n=1}^{N}\frac{\sinh(u-\lambda_n-r)\sinh(u-\lambda_n)\sinh(u+\lambda_n-r)}{\sinh(u+\lambda_n)\sinh(u-\lambda_n)}. \tag{21}$$

The eigenvalue $\alpha(u)$ and $d(u)$ turn out to be

$$\alpha(u) = \sinh(u+\alpha)\prod_{n=1}^{N}(p^{-1}q)^{n}\,2\sinh(u+\varepsilon_n r)\sinh(-u+\varepsilon_n r)$$

$$d(u) = 8\sinh(2u)\sinh(u+\alpha)\prod_{n=1}^{N}(p^{-1}q)^{\varepsilon_n}\sinh(u+\varepsilon_n r)\sinh(-u+\varepsilon_n r) + (q-p^{-1})\prod_{n=1}^{N}(p^{-1}q)^{\varepsilon_n}\sinh(u-\varepsilon_n r)\sinh(-u-\varepsilon_n r) \tag{22}$$

ε_n being the eigenvalue of H_n.

The Hamiltonian of the chain can be written as

$$H = \sum_{n=1}^{N-1} H_{n,n+1} + \frac{1}{2}K_-(0) + \frac{\mathrm{tr}_0 K_+(O)H_{NO}}{\mathrm{tr}\,K_+(O)} \tag{23}$$

where

$$H_{n'n+1} = P_{n,n+1}\frac{\mathrm{d}}{\mathrm{d}u}R_{n,n+1}(u)\bigg|_{u=0}.$$

Explicit computation leads to

$$H=\sum_{n=1}^{N-1}\left[\left(\frac{g-p^{-1}}{2}\right)\sigma_n^3\otimes\sigma_{n+1}^3+\frac{1}{2}(p^{-1}q+1)(\sigma_n^1\otimes\sigma_{n+1}^1+\sigma_n^2\otimes\sigma_{n+1}^2)\right.$$
$$\left.+\tfrac{1}{2}(p^{-1}q-1)(\sigma_n^2\otimes\sigma_{n+1}^1-\sigma_n^1\otimes\sigma_{n+1}^2)\right]+\frac{q+p^{-1}}{2}\sigma_N^3$$
$$+2\cosh\xi_+\cosh(r/2)\sigma_1^3. \tag{24}$$

4. Discussions

It is interesting to note that if $p=q$ then the present Hamiltonian reduces to that of Sklyanin. Also the Hamiltonian (24) leads to a new type of nearest neighbour interaction term $\sigma_n^2\otimes\sigma_{n+1}-\sigma_n^1\otimes\sigma_{n+1}^2$ which was a set in [9]. The R-matrix given in equation (3a) has a new feature that it describes a different vertex-type model instead of the usual six-vertex one, in conformity with the structure of the Hamiltonian given in (24). We now make an important observation. For $p=q$, the R matrix of equation (3a) reduces to that of [9] or to that of the usual six-vertex model, which can be written as

$$R=\begin{bmatrix} a & 0 & 0 & 0\\ 0 & b & c & 0\\ 0 & c & b & 0\\ 0 & 0 & 0 & a\end{bmatrix}$$

where $a=\sinh(u+r)$, $b=\sinh u$, $c=\sinh r$.

Now we make a gauge transformation

$$R\to R=K_{12}RK_{21}^{-1}$$
$$=\{1\otimes K\}R\{K^{-1}\otimes 1\}$$

where

$$K=\begin{pmatrix} P_1 & 0\\ 0 & P_2\end{pmatrix}$$

with $K_{12}=1\otimes K$ and $e^{2r}=pq$, $p_1^2=p^{-1}q$ and $p_2^2=1$.

Then an easy computation leads to

$$\tilde{R}=\begin{bmatrix} a & 0 & 0 & 0\\ 0 & b_1 & c & 0\\ 0 & c & b_2 & 0\\ 0 & 0 & 0 & a\end{bmatrix}$$

where

$$\tilde{a}=e^u q-e^{-u}p^{-1} \qquad \tilde{b}_1=e^u-e^{-u}$$
$$\tilde{b}_2=(e^u-e^{-u})p^{-1}q \qquad \tilde{c}=q-p^{-1}$$

which is the R-matrix obtained from the $SU_{p,q}(2)$ invariant spin chain.

Actually these type of gauge transformations were studied by de Vega and Akutsu *et al* for the construction of spin chain system with inhomogeneity. On the other hand the Hamiltonian given in equation (24) is also a variant of the *xxz* Hamiltonian. In particular we know

$$H_{xxz} = \sum h_{n,n+1}$$
$$h_{n,n+1} = \cosh r[\sigma_n^3 \otimes \sigma_{n+1}^3] + \sigma_n^1 \otimes \sigma_{n+1}^1 + \sigma_n^2 \otimes \sigma_{n+1}^2 .$$

Under the above transformation we observe that

$$\begin{aligned}\tilde{H} = K(H)K^{-1} \qquad & \text{with } K = \prod_n K_{n+1,n} \\ = \sum \tilde{h}_{n,n+1} & \\ = \sum [(p-q^{-1})\sigma_n^3 \otimes \sigma_{n+1}^3 & + (p^{-1}q+1)(\sigma_n^1 \otimes \sigma_{n+1}^1 + \sigma_n^2 \otimes \sigma_{n+1}^2) \\ + \tfrac{1}{2}(p^{-1}q-1)(\sigma_n^2 \otimes \sigma_{n+1}^1 & - \sigma_n^1 \otimes \sigma_{n+1}^2)\end{aligned}$$

which is the Hamiltonian given in (24) with a new type of interaction, except for the boundary terms.

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